Nim in Topological Spaces

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Contents

List of Figures

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1 Introduction and Background

We begin with some formal consideration of mathematical games.

1.1 Games

Informally, a mathematical *game* is “a game whose rules, strategies and outcomes are defined by clear mathematical parameters” [1]. Following the perspective of combinatorial game theory, we will largely be concerned with games that

1. are *positional*: the game consists of states called *positions*, so that gameplay consists of the players making *moves* between positions.

2. are *sequential*: each player chooses their action before others choose theirs, so that the others have knowledge of the move when making their own decision. The game may be seen as unfolding linearly in time.

3. are *impartial*: allowable moves depend only upon the current position, not upon the player currently moving. For example, chess is not impartial: at any given board configuration, white may only move white pieces and black only black.

4. involve *perfect information*: each player possesses the same knowledge of prior positions and moves. Chess exhibits perfect information, but poker does not.

We maintain that any game must have a winner: there must be some criterion for calling an end to the game as well as for declaring a winner. If a game is certain to end in a finite number of moves, we call it a *finite* game—though, as we shall see, not all interesting games are finite. Typically, a finite positional game ends when one of possibly many designated *terminal* positions is reached. Combinatorial game theory typically formulates games in such a way that the terminal positions are exactly those at which no further moves remain. Under *normal play convention*, the winner is the player who moves to a terminal position—in other words, you win by making the last move and if you can't move, you lose. *Misére* play indicates the opposite: one wins by forcing one's opponent into making the last move. [3]

Example 1. *Nim* is a finite positional sequential impartial game of perfect information for two players—we will refer to Players I and II, where I has the first move. Variants of the game date to ancient times, but the name "nim" was given by Bouton in 1901 [2], who developed a complete theory of the game. The game begins with finitely many piles of (finitely many) counters. A play consists of selecting a given pile and removing some number, possibly all, of the counters in the pile. The game ends when the terminal position, where no counters remain, is reached, and in normal play convention, the player who takes the last counter—plays the last move—wins. However, misére play, in which the player who takes the last counter loses, seems to be more widespread in actual practice.

Bouton identified a subset of positions he called *safe combinations*, which we will refer to as *balanced* positions, demonstrating that 1) any move made from a balanced position would afterward leave an unbalanced position, and 2) from any unbalanced position there is always a move that will leave a balanced position. In this way, the first player to play a balanced position may continue to do so on each of their successive turns while also forcing their opponent to play only unbalanced positions. In this analysis, the terminal position is balanced, so that a player executing the above strategy can ensure that their opponent will never be able to play it. Thus, under normal play convention the first player who can leave a balanced position possesses a *winning strategy*—a deterministic process for making moves which, when followed, always results in a win. If the initial position is unbalanced, Player I has a winning strategy; if the initial position is balanced, Player II does. We say that nim is a *determined* game—in any instance, one player has a winning strategy. Bouton also extended his analysis to demonstrate that misére play is determined.

The analysis of normal play nim proceeds by considering the binary representations of the sizes of the piles. For example, three piles of sizes 11, 2, and 7 are respectively represented by 1011, 10, and 111.

Definition 2. The *nim-sum* of numbers is given by place-wise binary addition. That is, the digit in position n (from the right) in the binary representation of the nim-sum is the sum modulo 2 of the digits in position n of the numbers being summed.

Thus the nim-sum of the numbers 11, 2, and 7—1011, 10, and 111—is 1110, or 14. It is helpful to write the summands above one another, with places aligned in columns.

$$\begin{array}{cccc} 1 & 0 & 1 & 1 \\ & & 1 & 0 \\ & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 \end{array}$$

A balanced position is now defined to be one for which the nim-sum of pile sizes is zero—every column has an even number of ones. We observe that, given the nim-sum of n numbers, knowledge of any $n-1$ of the numbers uniquely determines the other. In a game of nim on three piles, then, for any two piles of given size there is a unique third pile size resulting in a balanced position. The effect of this is that any move on a balanced position leaves an unbalanced position, since it leaves exactly two of three pile sizes unchanged. This is Bouton's first principle: balanced always goes to unbalanced.

To demonstrate his second principle, that there is always a move that balances an unbalanced position, Bouton gives a procedure for determining such a move. Given an unbalanced position, we may move on any pile whose representation contains a one in the leftmost unbalanced column. There is an odd number of ones in this column, so certainly such a pile exists. To decide how many rocks to leave remaining in this pile, we modify its current representation to create a new binary number: we change the one in the aforementioned column to a zero, then adjust all digits to the right as needed to balance the remaining unbalanced columns. To see that the new number is less than the old, we note that changing the selected one to a zero subtracts 2^n, for some n, while any change to the digits to the right can at most add back $2^n - 1$ (in the case where all such digits change from zero to one). In the example above, we locate the 8's column and have only the option to move on the first pile. We obtain the number 0101, or 5, which tells us to remove 6 rocks to leave 5.

$$\begin{array}{cccc} \mathit{0} & \mathit{1} & \mathit{0} & \mathit{1} \\ \cancel{1} & \cancel{0} & \cancel{1} & 1 \\ & & 1 & 0 \\ & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 \end{array}$$

In the special case of nim on two piles, a balanced position is one where the piles have equal size. The strategy above reduces essentially to copying the opponent's move: the player who plays balanced takes the same number of rocks their opponent did on the turn prior, but from the opposite pile, which was left larger. The opponent will be first to empty either of the two piles, leaving the balanced player to win on the next move by taking all of the remaining pile.

Definition 3. The *nimbers*, denoted $*0$, $*1$, $*2$, ..., represent values of a single nim pile: the nimber $*n$ is interpreted as the position in a game of nim where there is exactly one pile of size n. More formally, the nimbers are the ordinal numbers endowed with nim-sum addition and a multiplication operation known as *mex*.

Definition 4. The *sum of games* (or of positions in a game) is a way to talk about combining games, or playing many at once. The sum is itself a game, in which a move consists of a move on exactly one of the summands. Two games G and H are said to be *equivalent* if, for any game K, the same player has a winning strategy in both sums $G + K$ and $H + K$. In particular, this means the same player has a winning strategy in both G and H.

Theorem 5. *(Sprague-Grundy) Any impartial game under normal play convention is equivalent to a nimber.*

This theorem encapsulates a broader theory of games developed by its eponyms. Because nonzero nimbers are determined in favor of Player I and $*0$ in favor of Player II (since Player I, unable to move, loses), Player I has a winning strategy in an impartial game if and only if its Grundy number is nonzero. Thus a consequence of this theorem is that any impartial game is determined. Additionally, we have a sort of homomorphism from games to Grundy values: the Grundy value of a sum of games is the nim-sum of the games' individual Grundy values. When the games being added are nimbers, this rephrases the analysis given by Bouton, but, more generally, a sum of games can be thought of a game of nim on some number of piles. In this way, nim is paradigmatic of all impartial games.

Now that we have an initial understanding of nim, we'll introduce the basic machinery of topology. We assume a familiarity with elementary concepts of set theory.

1.2 Topological Spaces

Definition 6. A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X such that the following properties hold:

1) $\emptyset$ and X and are in $\mathcal{T}$

2) The union of any elements of $\mathcal{T}$ is in $\mathcal{T}$

3) The intersection of finitely many elements of $\mathcal{T}$ is in $\mathcal{T}$

A set X for which a topology has been specified is referred to as the *topological space* $(X, \mathcal{T})$, or merely the space X when context is understood. Members of $\mathcal{T}$ are called *open* sets, and their complements are called *closed*. [4]

Definition 7. For a point x in a space X, an open subset U is said to be an *open neighborhood* of x if $x \in U$; we say a sequence (x_n) *converges to* x in a space X if and only if every open neighborhood of x also contains x_n for all but finitely many n.

Definition 8. A *basis* for a topology on a set X is a collection $\mathcal{B}$ of subsets of X (called *basis elements*) such that the following properties hold:

1. For every $x \in X$ there is a basis element B with $x \in B$

2. For basis elements B_1, B_2 and any $x \in B_1 \cap B_2$, there is a basis element B_3 with $x \in B_3 \subseteq B_1 \cap B_2$

For a given basis $\mathcal{B}$ on a set X, we define the *topology* $\mathcal{T}$ *generated by* $\mathcal{B}$ by declaring a subset U of X to be open if for every $x \in U$ there is a basis element B with $x \in B \subseteq U$. Equivalently, the open sets of $\mathcal{T}$ are the arbitrary unions of basis elements. Note that basis elements are themselves open sets in the topology they generate. [4]

Definition 9. If for topologies $\mathcal{T}_1, \mathcal{T}_2$ we have that $\mathcal{T}_1 \subseteq \mathcal{T}_2$, we say that $\mathcal{T}_2$ is *at least as fine* as $\mathcal{T}_1$. If $\mathcal{T}_2 \subseteq \mathcal{T}_1$, we say that $\mathcal{T}_2$ is *at least as coarse* as $\mathcal{T}_1$. If the inclusion is proper, we say that a topology is *finer* or *coarser* respectively. Informally, a finer topology has "more" open sets and a coarser topology has "fewer". If neither inclusion holds, we say the topologies are *incomparable*.

Now we give some fundamental examples of topologies.

Example 10. On any set X, the *indiscrete* topology is given by $\mathcal{T}_i = \{\emptyset, X\}$, and the *discrete* topology is given by $\mathcal{T}_d = \mathcal{P}(X)$, the power set of X. Since $\mathcal{T}_i \subseteq \mathcal{T} \subseteq \mathcal{T}_d$ for any topology $\mathcal{T}$, the indiscrete is the coarsest possible topology and the discrete is the finest.

Example 11. The *standard* or *euclidean* topology on the real line is that generated by taking as basis all intervals of the form (a, b). The *lower limit* topology on the real line is that generated by a basis of all intervals of the form $[a, b)$. Since $(a, b) = \bigcup_{a<\alpha<b}[\alpha, b)$, the lower limit topology is finer than the standard. The two spaces possess contrasting properties: while the standard real line $\mathbb{R}$ is *connected*—it cannot be expressed as a disjoint union of two open sets—the lower limit line $\mathbb{R}_l$ is disconnected, since for any real number a we have $\mathbb{R}_l = (-\infty, a) \cup [a, \infty)$. In fact, $\mathbb{R}_l$ is *totally disconnected*—the only connected subsets are singletons. Further, fewer sequences converge in $\mathbb{R}_l$. Since $\mathbb{R}_l$ is finer than $\mathbb{R}$ and $[a, a + \epsilon)$ is an open neighborhood of any $a \in \mathbb{R}_l$, the convergent sequences (a_n) in $\mathbb{R}_l$ are exactly those which converge in $\mathbb{R}$ and for which $a_n \geq a$ for all but finitely many n. Thus, for example, the sequence $(-\frac{1}{n})_{n\in\mathbb{N}}$ converges in $\mathbb{R}$ but not in $\mathbb{R}_l$.

Once we have topologies, we can play games with them. We begin by outlining the framework for topological games given by Telgarsky [7].

1.3 Topological Games

Let G denote a game. We consider games of two players, Player I and Player II, where, as before, Player I starts play by making the first move. We say a *play* of a game is a sequence of moves of type ω, where a *move* specifies an object particular to each game (eg. players may be choosing points in a space, or subsets or covers). We may think of a position as a finite string of moves. Each play must result in a win for exactly one player and a loss for the other. We define a *strategy* for Player II to be a function F from the set of legal finite sequences (A_n) of moves of Player I to the set of moves of Player II such that $F((A_n))$ is legal from position A_n; a strategy for Player I is defined similarly. If a strategy for Player I has the property that Player I wins any play in which their moves are the outputs of the strategy function, that strategy is said to be a *winning strategy*.

It is convention to denote by $G(X)$ the game G played on a space X. Topological games are generally described for arbitrary spaces, but a particular choice of space may alter the existence of winning strategies—in fact, such existence or nonexistence often serves as a useful characterization of topological properties. One often sees results of the following form: "the class of spaces X for which Player I has a winning strategy in $G(X)$ is exactly the class of spaces possessing property P". If Player I has a winning strategy in $G(X)$, we may write $I \uparrow G(X)$, and similarly for Player II. If $I \uparrow G(X)$ or $II \uparrow G(X)$, we say that $G(X)$ is *determined*. We say two games G and H are *equivalent* when, for any space X, we have that $I \uparrow G(X)$ if and only if $I \uparrow H(X)$ and $II \uparrow G(X)$ if and only if $II \uparrow H(X)$. A *stationary* strategy for a player is a strategy that depends only upon the opponent's last move; a *Markov* strategy is one that depends only on the opponent's last move and its ordinal number. [7]

We introduce some additional topological concepts before describing two topological games, including one of the earliest examples.

Definition 12. Let X be a subset of a topological space Y.

- The *closure* of X in Y, denoted $\overline{X}$, is the set of all points of Y whose every open neighborhood has nonempty intersection with X—in a sense, points that cannot be "separated" from X by an open set. Eqivalently, $\overline{X}$ is the intersection of all closed sets containing X. Note that $X \subseteq \overline{X}$.

- The *interior* of X in Y, denoted X^o, is the collection of all points of X contained in an open neighborhood itself contained in X, or, equivalently, the union of all open sets contained in X. Note that $X^o \subseteq X$.

- X is said to be *dense* in Y if every open set of Y contains a point of X, or, equivalently, $\overline{X} = Y$.

Definition 13. Let X be a subset of a topological space Y. Then X is said to be

- *nowhere dense* if its closure has empty interior.

- *meager* or *first category* if it is the countable union of nowhere dense subsets

- *nonmeager* or *second category* if it is not meager.

A subset with meager complement is said to be *comeager* or *residual.*

Definition 14. A topological space Y is said to be *Baire* if, given any countable collection $\{A_n\}$ of closed subsets, each of which has empty interior, their union $\bigcup A_n$ also has empty interior. Equivalently, Y is Baire if the countable intersection of open sets, each of which is dense, is dense. [4]

Observation 15. A space Y is Baire if and only if every nonempty open set in Y is nonmeager.

Example 16. The *Banach-Mazur Game* is the first infinite positional game of perfect information studied by mathematicians [7]. Proposed by Mazur in 1935, the original formulation involves two players who, given a subset X of the closed unit interval J, alternately select subintervals of J to form a nested sequence $J_0 \supseteq J_1 \supseteq J_2 \ldots$ where the even-indexed subintervals are choices of Player I and the odd-indexed are choices of Player II. Player I wins if the intersection of all intervals J_n contains a point of X—that is, if $X \cap (\bigcap_{n<\omega} J_n) \neq \emptyset$—while Player II wins otherwise. We denote this game by $MB(X, J)$. Mazur observed two sufficient conditions for the game to be determined: if X is meager in J, then $II \uparrow MB(X, J)$, while if X has meager complement in some subinterval of J then $I \uparrow MB(X, J)$. He further posed the question of whether these conditions were necessary for the existence of the players' respective winning strategies.

Banach gave an affirmative answer shortly thereafter, though the first published proof was due to Oxtoby in 1957, who considered the game in more general terms, for a subset X of a topological space Y and any family $\mathcal{W}$ of subsets of Y such that

1. Every $W \in \mathcal{W}$ contains a nonempty open set of Y (has nonempty interior)
2. Every open set of Y contains an element W of $\mathcal{W}$.

We denote this game $MB(X, Y, \mathcal{W})$; letting Y be the unit interval J and $\mathcal{W}$ the subintervals of J, we obtain the original game $MB(X, J)$. Again, players take turns choosing subsets $W_0, W_1, W_2 \ldots$ from $\mathcal{W}$ with $W_0 \supseteq W_1 \supseteq W_2 \ldots$; Player I wins if and only if $X \cap (\bigcap_{n<\omega} W_n) \neq \emptyset$. Oxtoby demonstrated that $II \uparrow MB(X, Y, \mathcal{W})$ if and only if X is meager in Y and, with the assumption that Y is a complete metric space, $I \uparrow MB(X, Y, \mathcal{W})$ if and only if X is comeager in some open subset of Y.

Choquet introduced a further modification in a study on "siftable" spaces. Let $BM(X)$ denote the game $MB(X, Y, \mathcal{W})$ with $X = Y$, $\mathcal{W}$ = all nonempty open sets, and let Player II win exactly when $\bigcap W_n \neq \emptyset$. The space X is Baire if and only if Player I has no winning strategy [7].

Proposition 17. *Mazur's First Observation: If X is meager in J, then $II \uparrow MB(X, J)$*

The following explanation elaborates what is found in Oxtoby [5]. We first consider the special case when X is nowhere dense. Instead of $J_0, J_1, J_2 \ldots$ write $A_0, B_0, A_1, B_1 \ldots$ so that Player I's moves are As and Player II's are Bs, and the index denotes a move and its immediate response. Player II can pursue a winning strategy as follows:

- Player I chooses a subinterval $A_0 \subseteq J$. If the interval is not open, we may take its interior, an open interval.

- X is nowhere dense in J, so $(\overline{X})^o = \emptyset$ and any open set contains a point of $(\overline{X})^c$, so there is $a_0 \in A_0$ such that $a_0 \notin \overline{X}$. Then there is an open neighborhood U_0 of a_0 such that $U_0 \cap X = \emptyset$. $U_0 \cap A_0$ is an open neighborhood of a_0 contained in A_0; since the topology is generated by a basis of open intervals, $U_0 \cap A_0$ contains an open interval B_0 such that $a_0 \in B_0 \subseteq U_0 \cap A_0$. Player II plays B_0, and since $A_0 \cap B_0 \cap \ldots \subseteq B_0 \subseteq U_0 \subseteq X^c$, Player II has already won.

Now consider when X is meager, so that $X = \bigcup_{n \in \mathbb{N}} X_n$ where X_n is nowhere dense for each n. Any open interval contains a point in $(\overline{X_n})^c$ for any n, and Player II's strategy is at each stage to move to a subinterval disjoint from another of the X_n's:

- Player I chooses a subinterval $A_0 \subseteq J$. We may assume it's open.
- By the same argument as before, applied to X_0 instead of X, there is $a_0 \in A_0$ with open neighborhood U_0 satisfying $U_0 \cap X_0 = \emptyset$. Then $U_0 \cap A_0$ contains open interval B_0. Player II plays B_0. Note $B_0 \subseteq {X_0}^c$.
- Player I plays $A_1 \subseteq B_0$.
- Player II plays $B_1 \subseteq A_1$ such that $B_1 \subseteq {X_1}^c$.

  $\vdots$
- Player I plays $A_n \subseteq B_{n-1}$.
- Player II plays $B_n \subseteq A_n$ with $B_n \subseteq {X_n}^c$.

  $\vdots$

Since $A_0 \cap B_0 \cap A_1 \cap B_1 \ldots \subseteq B_n \subseteq {X_n}^c$ for every n, the infinite intersection cannot contain a point of X, and Player II wins.

At first glance, this appears to be a Markov, rather than stationary, strategy for Player II, who, in order to move to a subinterval excluding the appropriate X_n, apparently needs to know the index of the interval Player I played immediately prior. However, a stationary strategy is possible: Player II can still ensure the exclusion of every X_n from the infinite intersection by determining the least n for which $X_n \cap A_n \neq \emptyset$ and choosing B_n disjoint from that X_n. This latter approach only requires knowledge of Player I's prior move, not its ordinal number.

Proposition 18. *Mazur's Second Observation: If X is comeager in a subinterval of J then $I \uparrow MB(X, J)$.*

Player I has the following strategy:

- Player I plays an interval A_0 in which X is comeager.

- Player II plays $B_0 \subseteq A_0$.

- $X^c = Z = \bigcup_{n\in\mathbb{N}} Z_n$, where each Z_n is nowhere dense. On turn n, as in Player II's strategy above, Player I chooses a subinterval $A_n \subseteq B_{n-1} \cap {Z_{n-1}}^c$, with the additional condition that $\overline{A_n} \subseteq B_{n-1}$. This is

- Since $A_0 \supseteq B_0 \supseteq \overline{A_1} \supseteq A_1 \supseteq B_1 \supseteq \overline{A_2} \supseteq A_2 \ldots$, we have $A_0 \cap B_0 \cap A_1 \cap B_1 \ldots = \bigcap \overline{A_n}$. This intersection is a nested decreasing sequence of compact sets, and so contains a point. Since, as above, $(A_0 \cap B_0 \cap \ldots) \bigcap Z = \emptyset$, this is a point of X.

Now let's look at a second topological game, considered in Scheepers' *Combinatorics of Open Covers* [6].

Definition 19. Let X be a space. A collection $\mathcal{U}$ of sets whose union is X—that is, $\bigcup_{U\in\mathcal{U}} U = X$—is called a *cover* of X. If the members of $\mathcal{U}$ are open sets in the topology on X, we say $\mathcal{U}$ is an open cover.

Definition 20. Let X be a space. We say that X is *compact* if for any open cover $\mathcal{U}$ of X there is a finite subcollection $\mathcal{V}$ of sets belonging to $\mathcal{U}$ that is also a cover of X.

We may talk about covers and compactness of a subset of a topological space analogously. One of the first examples usually given of a compact subset of $\mathbb{R}$ is the closed interval $[a, b]$. In contrast, the open interval (a, b) is not compact, witnessed by the open cover $\mathcal{U} = \{(\alpha, b) \mid a < \alpha < b\}$ which has no finite subcover. Compactness is a fundamental property in topology and analysis, not without its own generalizations.

Definition 21. A space X is said to possess the *Menger* property if for every sequence $(\mathcal{U}_n)$ of open covers of X there is a sequence $(\mathcal{V}_n)$ such that $\mathcal{V}_n$ is a finite subcollection of $\mathcal{U}_n$ for each n and the collection $\bigcup_{n=1}^{\infty} \mathcal{V}_n$ covers X.

The Menger property is a strengthening of the *Lindelöf* property which asserts that every open cover has a countable subcover, itself a weakening of compactness. Indeed, letting $(\mathcal{U}_n)$ be a constant sequence at an arbitrary cover demonstrates this fact. The Menger property can be characterized by strategies in an infinite topological game.

Example 22. The *Menger Game* on a space X has two players: in round n, Player I chooses an open cover $\mathcal{U}_n$ of X and Player II chooses a finite subcollection $\mathcal{V}_n$ of $\mathcal{U}_n$. Player II wins the game if and only if $\bigcup_{n=1}^{\infty} \mathcal{V}_n$ covers X.

If Player II has a winning strategy in the Menger game on X, then X is Menger: we simply let Player I play the terms in an arbirtrary sequence $(\mathcal{U}_n)$ and the moves given by the strategy of Player II generate the desired cover. Though we have not found any conclusive information on the question, it seems likely that the converse does not hold. Scheepers [6], building on work of Hurewicz, showed that a space X is Menger if and only if Player I does not have a winning strategy in the Menger game on X; the nonexistence of a winning strategy for Player I is not in general equivalent to a winning strategy for Player II; we would need this equivalence in order to obtain the converse.

2 Nim in Topological Spaces

2.1 Difficulties in Adapting the Game

In developing a topological game that might sensibly be compared to nim, we faced an open ended task without one perfect answer. We sought a formulation that would reduce to classic combinatorial nim under some restrictions but would otherwise allow the notions of space and nearness characteristic of topology to carry gameplay and strategy towards interesting and unforeseen ends. The most immediate task was to decide what stones would represent and what piles would represent. What kind of existing topological structures organize discrete, atomic units into various groupings? Points might be taken for stones and open sets for piles; another possibility would take open sets for stones and collections of open sets for piles.

Finite topological spaces are not typically seen as characteristically topological—spaces with infinitely many points and open sets are most common in classic applications. Indeed, gameplay in infinite space appeared as one of the most naturally intriguing possibilities to explore in our adaptation. However, we had to be able to assign a winner to a round of our game. If in nim the game is guaranteed to end by the fact that every decreasing sequence of natural numbers ends—you only have so many rocks to choose—it would be conceivable for the players in our game to keep choosing rocks endlessly without exhausting a pile, or to keep exhausting piles without finishing all of infinitely many piles. It would be necessary to introduce conditions to guarantee that a last rock is chosen, or to abstract the condition "the winner is the player who chooses the last rock" to a criterion for evaluating infinite sequences of moves like those deciding the Banach-Mazur or Menger games.

2.2 Cover Partition Game with a Partition of Size Two

The formulation we arrived at is as follows. Let X be a space, $\mathcal{C}$ a cover of X, and $\mathcal{P} = \{\mathcal{P}_i\}_{i \in I}$ a partition of $\mathcal{C}$. We may suppose C is a cover of open sets, although this is actually not neccesary for our results so far. For a given n, let $\mathcal{U}_n$ denote a specified subcollection of some $\mathcal{P}_i$ and let $\mathcal{V}_n$ denote $\bigcup_{j \leq n} \mathcal{U}_j$. Players alternately specify $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3 \ldots$ such that $\mathcal{U}_{n+1} \nsubseteq \mathcal{V}_n$. The game ends when $X \subseteq \bigcup_{V \in \mathcal{V}_n} V$. That is, players choose some covering sets (not already chosen) from a given partition and add them to a cumulative collection; the game ends when they have made a cover of X. We may think of each successive collection $\mathcal{V}_n$ of every covering set so far chosen as a position in our game. For now, we adopt a normal play convention by saying that the player who completes the cover is the winner.

To begin with a simple instance of our game, that would certainly guarantee an end, we first considered finite covers (and thus a partition with finitely many parts). In the case where $\mathcal{C}$ is *irreducible*—no proper subcover exists—the game is just like nim. Every element of $\mathcal{C}$ must be chosen before the game ends, and those elements are split into groups of differing sizes. The case where $\mathcal{C}$ is reducible—where nontrivial subcovers exist—introduces complexity, since, in a sense, one no longer needs to take all the rocks in order to win. If $|\mathcal{P}| = 1$, Player I can take everything in $\mathcal{P}_1 = \mathcal{C}$ on the first turn and immediately win. So we will first consider partitions of size two. We begin by applying a fundamental game theoretic concept.

2.3 A Lattice of P's and N's

Definition 23. In a mathematical game, a position is called an N *position* if the next player (the player whose move it is) has a winning strategy from that position onward. A position is called a P *position* if the previous player (the player who moved to the current position) has a winning strategy from thereon. These two categories are known as *outcome classes.* We say that a position *sees* another position if it is possible to reach the other in one move.

In a finite game, all positions can be classified as N or P positions. Any position that sees a P position is an N position, since the current player can move to the P

and enact some winning strategy. On the other hand, any position that sees only N positions is a P position, since, regardless of where the current player moves, their opponent starts their next turn with a winning strategy. Since these cases are complementary, a position can be classified as long as all of those it sees have been classified. Any unclassifiable position must see another unclassifiable position, which must see another, forming a chain of moves which must reach a terminal position which is also unclassifiable. But since terminal positions are necessarily P positions, this can't be. It is thus possible to work backwards from the terminal positions to assign a class to every position. In particular, this means that every finite game is determined by the class of the initial position.

How might we apply this framework to our game? A simple cover of size four, split into two parts, is shown in Figure 2.3. The possible positions in our game are simply the subcollections of this cover. Successive moves always respect the inclusion relation on subcollections—$\mathcal{V}_n \subseteq \mathcal{V}_{n+1}$—so the basic trajectory of positions is overlaid on the power set poset for the collection $\mathcal{C}$ of covering sets, represented in Figure 2. Nodes represent positions, with the lowest node indicating the starting position, where no sets have been chosen, and the highest indicating the position where every set from the cover $\mathcal{C}$ has been chosen. A red or blue line represents a move which adds one set from $\mathcal{P}_1$ or $\mathcal{P}_2$ respectively; thus a legal move may follow any path upward that consists entirely of a single color. Positions are organized into rows according to the number of sets they contain: for example, the widest row includes the six positions comprised of exactly two covering sets. Black coloring indicates the subcovers of $\mathcal{C}$, and the winner is the first to arrive at any black node. Unlike nim, our cover game does not possess a unique terminal position. With the knowledge that black nodes are P positions, we can work backwards to classify the remaining positions according

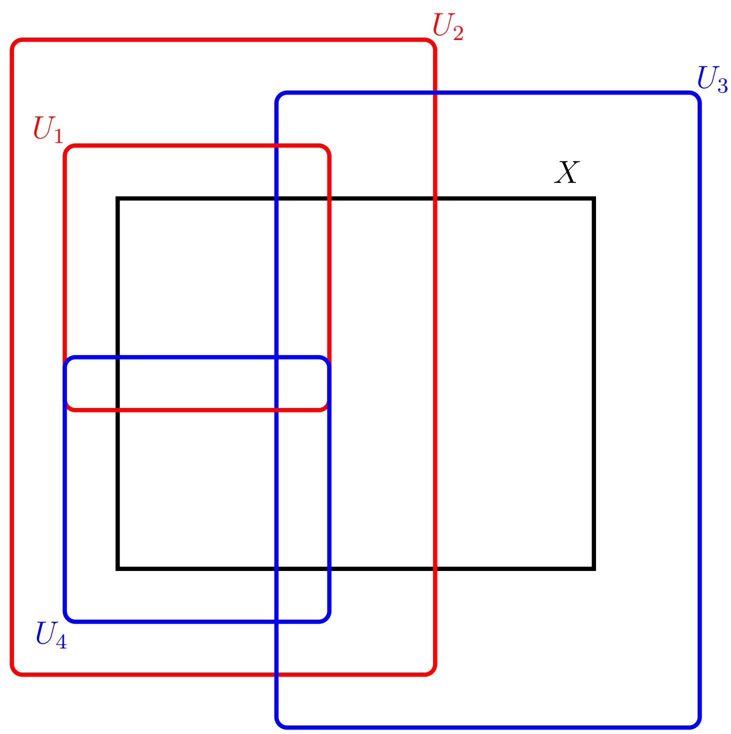

Figure 1: Four covering sets cover a space. $\mathcal{P}_1 = \{U_1, U_2\}$, $\mathcal{P}_2 = \{U_3, U_4\}$

to the logic given above. In this example, since the start position is an N, we find that the game is determined in favor of Player I.

2.4 Measure

In this section, we offer an original way to think about strategy for the game on two parts. We may, in a sense, treat one position as two: the sets chosen so far may be split into those taken from $\mathcal{P}_1$ and those from $\mathcal{P}_2$. We then have two poset structures, which may be considered independently. With $|\mathcal{P}| = 2$, let $X_1 = \bigcup_{U \in \mathcal{P}_1} U$, let $X_2 = \bigcup_{U \in \mathcal{P}_2} U$, let $Y_1 = X_1 \setminus X_2 = X \setminus X_2$ and $Y_2 = X_2 \setminus X_1 = X \setminus X_1$. That is, Y_1 consists of points not contained in any covering set of $\mathcal{P}_2$ (those exclusively contained in covering sets of $\mathcal{P}_1$) and Y_2 consists of points not contained in any covering set of

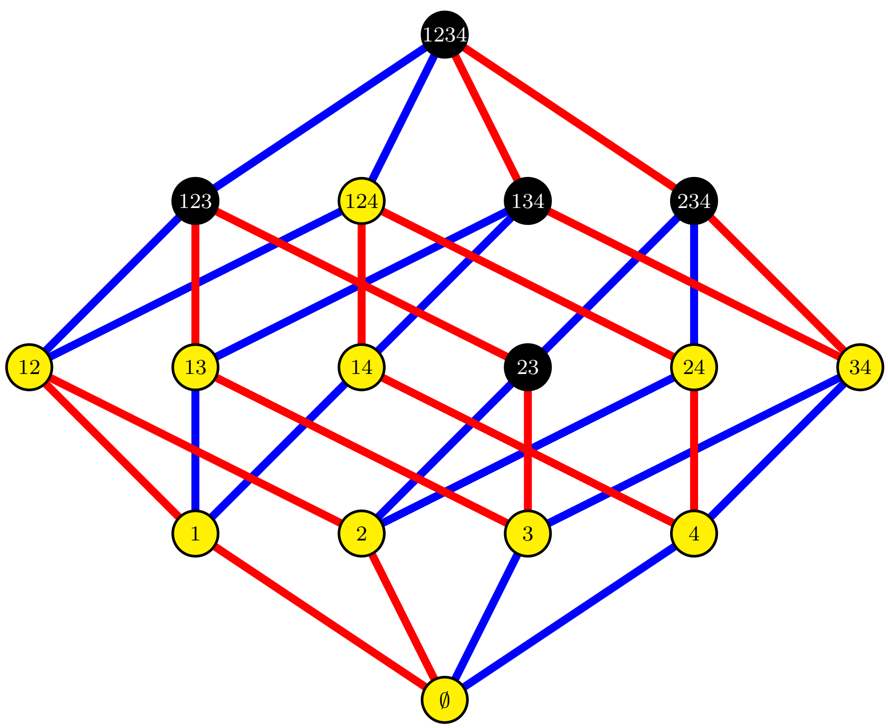

Figure 2: Positions in an instance of our game.

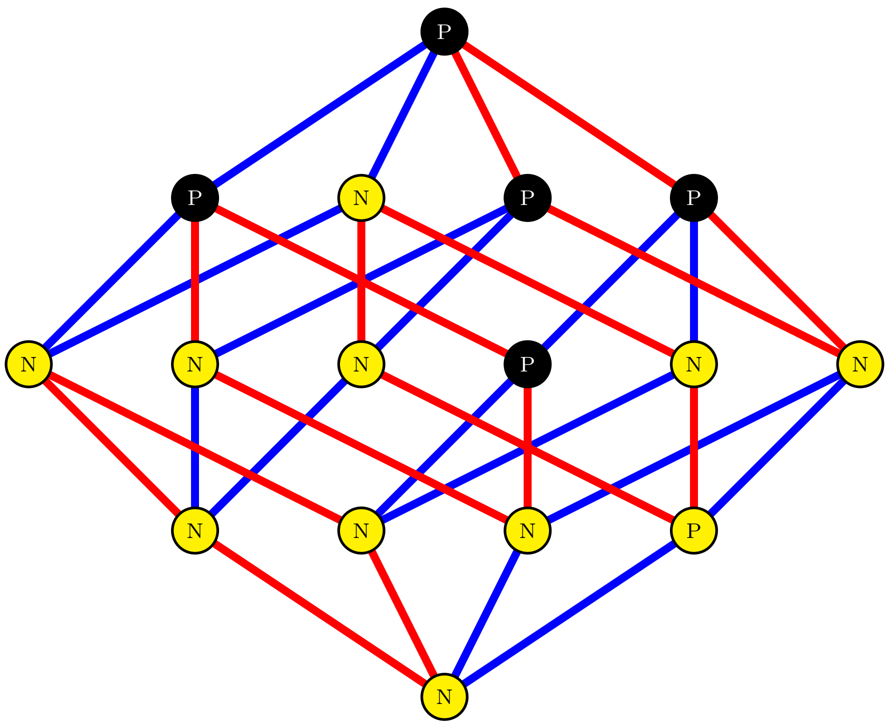

Figure 3: Positions classified.

$\mathcal{P}_1$. Whichever player is the first to cover one of Y_1 or Y_2 will lose, since the next player can win by taking everything left in $\mathcal{P}_2$ or $\mathcal{P}_1$ respectively. Thus, a strategy by which a player can force their opponent to first cover Y_1 or Y_2 would be a winning strategy. This would necessitate leaving one's opponent with a position in which choosing any additional covering set from $\mathcal{P}_1$ completes a cover of Y_1 and likewise for $\mathcal{P}_2$ and Y_2. Since $\mathcal{P}_1$ is finite and covers Y_1, it necessarily has at least one "maximal" subcollection which cannot be added to without covering Y_1, as does $\mathcal{P}_2$ respectively. Since the set of all subcollections of $\mathcal{P}_1$ is finite, any subcollection that does not cover Y_1 is contained in some maximal subcollection (else we could keep adding covering sets indefinitely without forming a cover). So we could try to ask how close a given noncovering subcollection is to becoming maximal, in order to track $\mathcal{V}_n \cap \mathcal{P}_1$ and $\mathcal{V}_n \cap \mathcal{P}_2$ as our game progresses. If both $\mathcal{V}_n \cap \mathcal{P}_1$ and $\mathcal{V}_n \cap \mathcal{P}_2$ are maximal, then the next move must cover Y_1 or Y_2 and lose. We introduce the following definition.

Definition 24. For a cover $\mathcal{C}$ of a space X, and a subcollection $\mathcal{N}$ of $\mathcal{C}$, we denote by $||\mathcal{N}||_X^{\mathcal{C}}$ (or $||\mathcal{N}||$ when context is clear) the smallest k such that adding any k other sets (sets in $\mathcal{N}^c$) in $\mathcal{C}$ to $\mathcal{N}$ covers X. This number $||\mathcal{N}||_X^{\mathcal{C}}$ will be called the *measure* of $\mathcal{N}$ *with respect to* X.

Equivalently, $||\mathcal{N}||$ is the smallest k where the union of any k other sets contains $(\bigcup_{N\in\mathcal{N}} N)^c$. When $\mathcal{C}$ is finite, well ordering of natural numbers guarantees such k exists, because the set of candidates is nonempty: any $|\mathcal{C}| - |\mathcal{N}|$ additional sets complete a cover. When $k \geq 1$, it follows that $k-1$ is largest such that there are $k-1$ additional covering sets that don't complete a cover, or don't contain $(\bigcup_{N\in\mathcal{N}} N)^c$.

It is immediate from this definition that a subcover $\mathcal{D}$ within $\mathcal{C}$ has measure $||\mathcal{D}||_X^{\mathcal{C}} = 0$, and a maximal subcollection $\mathcal{M}$ has measure $||\mathcal{M}||_X^{\mathcal{C}} = 1$. In our game, $||\mathcal{V}_n \cap \mathcal{P}_1||_{Y_1}^{\mathcal{P}_1}$

gives some measure of how close Y_1 is to being covered, and similarly $||\mathcal{V}_n \cap \mathcal{P}_2||_{Y_2}^{\mathcal{P}_2}$ for Y_2. Our goal is to show that these two statistics behave much like the sizes of piles in a game of nim on two piles. We prove the following propositions.

Proposition 25. *Let $\mathcal{N}$ and $\mathcal{A}$ be disjoint collections of covering sets drawn from a cover $\mathcal{C}$. Let $||\mathcal{N}|| = k$ and $|\mathcal{A}| = i$ where $0 \leq i \leq k$. Then $||\mathcal{N} \cup \mathcal{A}|| \leq k - i$.*

Proof. Let $\mathcal{B}$ be a collection of $k-i$ covering sets from $\mathcal{C}$ with $(\mathcal{N} \cup \mathcal{A}) \cap \mathcal{B} = \emptyset$. Since $|\mathcal{A} \cup \mathcal{B}| = k$ and $||\mathcal{N}|| = k$, $\mathcal{N} \cup (\mathcal{A} \cup \mathcal{B}) = (\mathcal{N} \cup \mathcal{A}) \cup \mathcal{B}$ forms a cover. Since $\mathcal{B}$ was arbitrary, $||\mathcal{N} \cup \mathcal{A}|| \leq k - i$. □

Proposition 26. *Let $\mathcal{N}$ be a collection of covering sets with $||\mathcal{N}|| = k$. For any i with $0 \leq i \leq k$, there is $\mathcal{A}$ such that $|\mathcal{A}| = i$ and $||\mathcal{N} \cup \mathcal{A}|| = k - i$.*

Proof. When $i = 0$, we may take $\mathcal{A}$ to be the empty collection. So assume $k \geq 1$. When $i = k$, we may take $\mathcal{A}$ to be any collection of size k disjoint from $\mathcal{N}$, so we consider $i \leq k - 1$. Since $||\mathcal{N}|| = k$ there is a collection $\mathcal{B}$ such that $|\mathcal{B}| = k - 1$, $\mathcal{N} \cap \mathcal{B} = \emptyset$, and $\mathcal{N} \cup \mathcal{B}$ does not form a cover. Let $\mathcal{A}$ be any subcollection of $\mathcal{B}$ of size i. Since $\mathcal{N}$ and $\mathcal{A}$ are disjoint, Proposition 25 gives $||\mathcal{N} \cup \mathcal{A}|| \leq k - i$. Now $\mathcal{B} \setminus \mathcal{A}$ has size $k - i - 1$ and $(\mathcal{N} \cup \mathcal{A}) \cap (\mathcal{B} \setminus \mathcal{A}) = \emptyset$ while $(\mathcal{N} \cup \mathcal{A}) \cup (\mathcal{B} \setminus \mathcal{A}) = \mathcal{N} \cup \mathcal{B}$ does not form a cover. Thus $||\mathcal{N} \cup \mathcal{A}|| \geq k - i$. Together, these inequalities give the result. □

It is worth noting that, while any $k-1$ sets that don't complete a cover make a maximal collection when added to to $\mathcal{N}$, there may be maximal collections comprising $\mathcal{N}$ and fewer than $k - 1$ additional sets. For the appropriate $\mathcal{A}$, the inequality of Proposition 25 may be strict.

Concerning our game, Proposition 25 says that choosing sets from $\mathcal{P}_1$ reduces the measure $||\mathcal{V}_n \cap \mathcal{P}_1||_{Y_1}^{\mathcal{P}_1}$, and likewise moving on $\mathcal{P}_2$ reduces $||\mathcal{V}_n \cap \mathcal{P}_2||_{Y_2}^{\mathcal{P}_2}$. Proposition 26 says that either of these measures can be reduced arbitrarily on a given turn. To pursue an analogy with nim, we call a position $\mathcal{V}_n$ *balanced* when $||\mathcal{V}_n \cap \mathcal{P}_1|| = ||\mathcal{V}_n \cap \mathcal{P}_2||$, and unbalanced otherwise. Then Proposition 25 says that if a position is balanced, any move makes the next position unbalanced, while Proposition 26 says that, if a position is unbalanced, there is a move making the next position balanced. Our game ends when both measures reach zero, as does nim when each pile is empty. So a winning strategy for the game on two parts is like that of nim on two piles, with Player I having a winning strategy if and only if the starting position is unbalanced and Player II having a winning strategy if and only if the start position is balanced.

Let's apply this perspective to the cover we considered in the previous section. Figure 2.4 reproduces the cover with the regions Y_1 and Y_2 shaded. To determine if the starting position is balanced or unbalanced, we want to find $||\emptyset||^{\mathcal{P}_1}_{Y_1}$ and $||\emptyset||^{\mathcal{P}_2}_{Y_2}$. Since Y_1 is contained in both U_1 and U_2, choosing any one set from $\mathcal{P}_1$ covers Y_1. However the same is not true for $\mathcal{P}_2$, since choosing U_4 does not complete a cover of Y_2. Thus the "pile sizes" at the start are 1 and 2. Our strategy says that Player 1 should move on $\mathcal{P}_2$, reducing its corresponding measure from 2 to 1; the only way to do this is to choose U_4. Then any move available to Player II would cover one of Y_1 or Y_2, allowing Player I to finish the game. In this case, Player I has the winning strategy—this confirms what we found in the lattice of positions previously.

2.5 Measure with "passes"

If our game is really about covering Y_1 and Y_2, we might only consider something a "move" if it adds new coverage to one of those subspaces. There may be certain covering sets in $\mathcal{C}$ which, taken individually, don't constitute a move in this sense. A

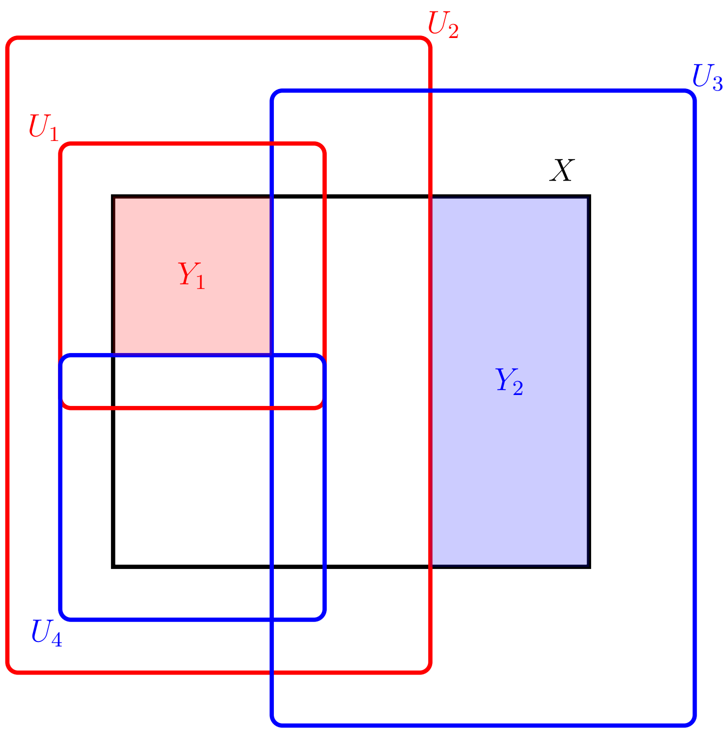

Figure 4: The example cover with Y_1 and Y_2 shaded.

first category is those $U \in \mathcal{C}$ for which $U \cap Y_1 = U \cap Y_2 = \emptyset$, i.e. if $U \subseteq X_1 \cap X_2$. A second concerns those U whose intersection with Y_1 or Y_2 is nontrivial, but which, on a given turn, is wholly contained in space already covered in prior turns–that is, on turn $n+1$, we have $\emptyset \neq U \cap Y_1 \subseteq \bigcup_{V \in \mathcal{V}_n} V$ or $\emptyset \neq U \cap Y_2 \subseteq \bigcup_{V \in \mathcal{V}_n} V$. Since $U \cap Y_1 \neq \emptyset$ implies $U \in \mathcal{P}_1$ and $U \cap Y_2 \neq \emptyset$ implies $U \in \mathcal{P}_2$, it is impossible for a covering set to have nontrivial intersection with both Y_1 and Y_2. A move $\mathcal{U}_{n+1}$ consisting entirely of covering sets from the above categories could intuitively be considered a "pass".

If we might for a moment ignore covering sets in the second category, considering only those in the first, we might consider excluding the latter from the collections whose measure we consider. We could isolate the subcollection $\mathcal{D}$ of $\mathcal{C}$ of covering sets contained in $X_1 \cap X_2$ by defining measures as before on subcollections of $\mathcal{P}_1 \setminus \mathcal{D}$ and $\mathcal{P}_2 \setminus \mathcal{D}$. As long as $\mathcal{D}$ is nonempty, a player would have the option to pass, effectively trading possession of a winning strategy: where before a given player had only the option to unbalance a balanced position, passing would force their opponent to play on such a position. These trades could continue back and forth until $\mathcal{D}$ is exhausted, when whoever has the winning strategy will retain it.

This suggests a way to describe a game different from but necessarily equivalent to the game we have considered so far—almost a game of nim inside nim, since sets in D are also divided into two parts. This game seems to pose combinatorial challenges on a level above those of basic nim, and so far we have not been able to develop a practical approach. In this rendering, one has four piles—$\mathcal{P}_1 \setminus \mathcal{D}$, $\mathcal{P}_1 \cap \mathcal{D}$, $\mathcal{P}_2 \setminus \mathcal{D}$, and $\mathcal{P}_2 \cap \mathcal{D}$—with the condition that the first pair and the latter pair may be moved on simultaneously. We have still more subtle challenges if we wish to take into account the sets in the second category, as their identity as passes depends on what has been selected prior and may change as the game progresses. One imagines the ability to add rocks back to a pile under certain circumstances.

However, we maintain that, though it may offer an interesting reframing of our problem, the behavior of these special covering sets does not pose a threat to our analysis of measure as presented so far. Our definition of measure already takes these cases into account. For example, we don't have to worry about a situation in which the collection $\mathcal{V}_n \cap \mathcal{P}_1$ is maximal "but" some set still remains in $\mathcal{P}_1 \cap \mathcal{D}$, belying the collection's maximality: any set in $\mathcal{P}_1 \cap \mathcal{D}$ is already contained in any maximal collection.

2.6 Cover Partition Game with a Partition of Size 3

In this section, we consider the difficulties of extending our analysis of measure to a game with three piles, with $|\mathcal{P}| = 3$ and $X_i = \bigcup_{U \in \mathcal{P}_i} U$ for $i = 1, 2, 3$. As above, a winning move must complete a cover of X by selecting covering sets from some $\mathcal{P}_i$, after a preceding move which left all uncovered space wholly contained in at least one X_i. Thus, assuming optimal play, the loser will be the first to cover one of the three regions $Y_{23} = X \setminus X_1$, $Y_{13} = X \setminus X_2$, and $Y_{12} = X \setminus X_3$. We might try to track how close a position is to covering each of these regions. On two parts, we measured subcollections of $\mathcal{P}_1$ with respect to Y_1, because $\mathcal{P}_2$ contributed nothing directly towards covering Y_1; applying the same reasoning, we would choose, for example, to measure subcollections of $\mathcal{P}_1 \cup \mathcal{P}_2$ with respect to Y_{12} because those are the parts containing sets which cover points in Y_{12}. We could define a subcollection to be maximal (with respect to Y_{12}) if adding any other covering set from $\mathcal{P}_1 \cup \mathcal{P}_2$ completes a cover of Y_{12}, and similarly for the regions Y_{13} and Y_{23}. Then we might imagine tracking the three measures as we track the size of three piles in a game of nim, with analogous strategy.

However, the comparison of measure to the size of nim piles becomes more tenuous—more precisely, it is difficult to define the proper measure to facilitate the comparison. It is problematic both that the collections $\mathcal{P}_1 \cup \mathcal{P}_2$, $\mathcal{P}_1 \cup \mathcal{P}_3$, and $\mathcal{P}_2 \cup \mathcal{P}_3$ share common elements and that they are comprised of more than a single part. The former property allows that a choice of sets from a given part will reduce more than one measure at once because two of the three measures measure collections drawn (partly) from that part. For example, choosing sets from $\mathcal{P}_2$ adds to the size of the subcollections being measured in $\mathcal{P}_1 \cup \mathcal{P}_2$ and $\mathcal{P}_2 \cup \mathcal{P}_3$ both, necessarily decreasing their measure (Prop 25). Now one is moving on two piles at once.

The latter property challenges the usefulness of Proposition 26, since even if, after move n, there exists a subcollection $\mathcal{U}_{n+1}$ of $\mathcal{P}_1 \cup \mathcal{P}_2$ that reduces $||\mathcal{V}_{n+1} \cap (\mathcal{P}_1 \cup \mathcal{P}_2)||_{Y_{12}}^{\mathcal{P}_1 \cup \mathcal{P}_2}$ by a desired amount, its members may be split between $\mathcal{P}_1$ and $\mathcal{P}_2$, making it an illegal choice. It is imaginable that, after some move, $||\mathcal{V}_n \cap (\mathcal{P}_1 \cup \mathcal{P}_2)|| = 8$ but the three collections of size 7 in $\mathcal{P}_1 \cup \mathcal{P}_2$ which do not complete a cover of Y_{12} possess 5 and 2, 3 and 4, and 4 and 3 sets from $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively. Proposition 25 guarantees the possibility of reducing the measure to 2 (by taking up to 5 from $\mathcal{P}_1$) but says nothing about the possibility of reducing to 1 or completing a cover. Now one is not free to choose any number of rocks from a pile.

The choice to measure coverage with respect to Y_{12}, Y_{13}, and Y_{23} also introduces the subtlety that the game may end after only one of these regions is covered—after only pile is reduced to zero. This contrasts with classic nim, where the winning move comes after a position with two of three piles—all but one—reduced to zero. We would at least need to consider if Bouton's original strategy could still be applied to a game resembling nim but with adjusted rules.

Even if we define each of our measures for collections taken from a single part, we encounter issues. With two parts, we ignored the region $X_1 \cap X_2$ since it could be covered if still necessary with the very last move. The remaining space to consider split neatly into regions covered by exactly one of X_1 or X_2. However, with three parts, after dismissing the triple intersection $X_1 \cap X_2 \cap X_3$, we still are obliged to measure the "flower petal" $P_{12} = (X_1 \cap X_2) \setminus X_3$ and its two analogues. But the measure $||\cdot||_{P_{12}}^{\mathcal{P}_1}$, for example, seems to miss part of the picture since sets from $\mathcal{P}_2$ also contribute to covering P_{12}. There are plenty of inventive ways to divide the space outside $X_1 \cap X_2 \cap X_3$, but all seem to face this same problem.

To get around this, instead of the complements of each X_i, we could consider the regions comprised of points in exactly one X_i (the complements of each union of a pair of X_i's). For a partition of size two, these choices result in the same two regions. But for size three the latter choice gives the three regions $Y_1 = X \setminus (X_2 \cup X_3)$, $Y_2 = X \setminus (X_1 \cup X_3)$, and $Y_3 = X \setminus (X_1 \cup X_2)$. Following this approach, we could measure subcollections of $\mathcal{P}_i$ with respect to Y_i. A move on one part then changes only one such measure, and each measure changes only when the relevant part is moved on. However, the problem arises that covering two (or even all) of these latter regions is not, at least on first glance, sufficient for the next player to win on the next move. We would like the player who reduces the three measures to 0-0-0 to be able to win with the same move, as is possible on two parts. But what if you go from 1-0-0 to 0-0-0, moving on $\mathcal{P}_1$ when $P_{23} = (X_2 \cap X_3) \setminus X_1$ is not fully covered?

Question 27. How do various possible measures relate to one another? Is there useful structure?

We demonstrate some cursory results.

Proposition 28. *Let $X = X_1 \cup X_2$ and let $\mathcal{C}$ be a cover of X. If $||\mathcal{N}||_{X_1}^{\mathcal{C}} = k_1$ and $||\mathcal{N}||_{X_2}^{\mathcal{C}} = k_2$, then $||\mathcal{N}||_X^{\mathcal{C}} = \max\{k_1, k_2\}$.*

Let $\max\{k_1, k_2\} = k$ and let $\mathcal{A}$ be a subcollection of $\mathcal{C}$ with $|\mathcal{A}| = k$ and $\mathcal{N} \cap \mathcal{A} = \emptyset$. Since $k \geq k_1$ and $k \geq k_2$, $\mathcal{N} \cup \mathcal{A}$ covers X_1 and X_2. Then $\mathcal{N} \cup \mathcal{A}$ covers X. Since $\mathcal{A}$ was arbitrary, $||\mathcal{N}||_X^{\mathcal{C}} \leq k$. Now suppose, without loss of generality, that $k_1 \geq k_2$. Since $||\mathcal{N}||_{X_1}^{\mathcal{C}} = k_1$, there is a subcollection $\mathcal{B}$ of $\mathcal{C}$ with $|\mathcal{B}| = k_1 - 1$ and $\mathcal{N} \cap \mathcal{B} = \emptyset$ such that $\mathcal{N} \cup \mathcal{B}$ does not cover X_1, and so does not cover X. Thus $||\mathcal{N}||_X^{\mathcal{C}} \geq k$.

Proposition 29. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be disjoint covers of X and let $\mathcal{N}_1 \subseteq \mathcal{C}_1$ and $\mathcal{N}_2 \subseteq \mathcal{C}_2$. If $||\mathcal{N}_1||_X^{\mathcal{C}_1} = k_1$ and $||\mathcal{N}_2||_X^{\mathcal{C}_2} = k_2$ then $||\mathcal{N}_1 \cup \mathcal{N}_2||_X^{\mathcal{C}_1 \cup \mathcal{C}_2} \leq k_1 + k_2 - 1$.*

Take any collection $\mathcal{A}$ in $\mathcal{C}_1 \cup \mathcal{C}_2$ with $(\mathcal{N}_1 \cup \mathcal{N}_2) \cap \mathcal{A} = \emptyset$ and $|\mathcal{A}| = k_1 + k_2 - 1 = (k_1 - 1) + (k_2 - 1) + 1$. By the pidgeonhole principle, either $\mathcal{A}$ contains k_1 covering sets from $\mathcal{C}_1$ or k_2 from $\mathcal{C}_2$. Then either $\mathcal{N}_1 \cup \mathcal{A}$ or $\mathcal{N}_2 \cup \mathcal{A}$ is a cover of X. Since $(\mathcal{N}_1 \cup \mathcal{N}_2) \cup \mathcal{A}$ contains both, it covers X. Since $\mathcal{A}$ was arbitrary, the result follows.

Question 30. If $\mathcal{N}_1, \mathcal{N}_2 \subseteq \mathcal{C}$ are disjoint, can we compare $||\mathcal{N}_1 \cup \mathcal{N}_2||_X^{\mathcal{C}}$ to $||\mathcal{N}_1||_X^{\mathcal{C}}$ and $||\mathcal{N}_2||_X^{\mathcal{C}}$?

3 Future Directions

3.1 Infinite Covers and Noetherian Spaces

To this point, we have restricted our attention to finite covers of a space. This condition clearly provided a guarantee that our game would end, but perhaps at too great a cost. Just as infinite spaces are generally seen as more characteristic than finite spaces, most important work with covers allows that an arbitrary cover might be infinite. As one point of reference, the work in Scheepers' *Combinatorics of Open Covers* [6] primarily concerns the infinitary combinatorics of infinite covers with various properties. Thus we would like to be able to play our game with an infinite cover. Are there other less restrictive ways to ensure a winner in our game that are still consistent with our intuition about how a game of nim should end?

In fact, the property we require of a cover $\mathcal{C}$ suitable for our game is not finiteness but, more precisely, that we can't keep adding covering sets forever—in the sequence $\mathcal{V}_1$, $\mathcal{V}_2$, $\ldots$, there must be some n for which $\mathcal{V}_n$ covers X. If we picture a region of coverage expanding as the game progresses, we would like to guarantee a turn where the region expands to cover all of X. The region of coverage on turn n is precisely the union of the sets contained in $\mathcal{V}_n$, and taking that union for each n results in a nested nondecreasing sequence of sets (since $\mathcal{V}_{n+1}$ only adds new sets to $\mathcal{V}_n$). This evokes the following concepts, already existing in the literature.

Definition 31. A poset P is said to have the *descending chain condition* if every strictly decreasing sequence $x_1 > x_2 > x_3 \ldots$ is finite, and the *ascending chain condition* if every strictly increasing sequence $x_1 < x_2 < x_3 \ldots$ is finite. A topological space is said to be *Noetherian* if its closed sets, ordered by inclusion, possess the descending chain condition, or, equivalently, if its open sets possess the ascending chain condition. That is, any sequence of open sets $U_1 \subseteq U_2 \subseteq \ldots$, where the inclusions are strict, must terminate—or, if the inclusions are not strict, then the sequence must be eventually constant.

Many examples of Noetherian spaces come from algebraic geometry, such as the

Zariski topology on the prime ideals of a commutative ring. We might try drawing our cover $\mathcal{C}$ from the open sets of a Noetherian space, allowing that $\mathcal{C}$ may be infinite. However, we must take into account that the ascending chain condition applies to strictly increasing chains. So far we have not made any stipulation that a move must reach some new area; it has been possible to choose a set that is wholly contained within the area currently covered. Thus it could be possible for the players to draw out the game indefinitely by making such moves. But if we could avoid this scenario, possibly by requiring moves to expand the region covered, we could ensure that our players create a cover in finitely many moves.

The following proposition is well known.

Proposition 32. *A space X is Noetherian if and only if every subset of X is compact.*

Proof. Let X be Noetherian and $\mathcal{U}$ an open cover of a subset Y of X. Let $U_1 \in \mathcal{U}$ and begin to choose U_{n+1} from $\mathcal{U}$ with $U_{n+1} \nsubseteq \bigcup_{i \leq n} U_i$. Then $(\bigcup_{i \leq n} U_i)_{n \in \mathbb{N}}$ is a strictly increasing sequence of open sets, and since X is Noetherian, there is some N for which it is impossible to choose U_{N+1} in the manner described. This means that every covering set in $\mathcal{U}$ is contained in $\bigcup_{i \leq N} U_i$. Since every point of Y is contained in some open set of $\mathcal{U}$, we have that $\{U_n \mid n \leq N\}$ is a finite subcover of Y.

Now instead suppose every subspace of X is compact. Let $U_1 \subseteq U_2 \subseteq \ldots$ be a sequence of open sets. Let $U = \bigcup_{n=1}^{\infty} U_n$. Since $\{U_n \mid n \geq 1\}$ is an open cover of U, which is compact, it has a finite subcover $\{U_n \mid n \leq N\}$. Then $U \subseteq \bigcup_{n=1}^{N} U_n = U_N$ and $U_N \subseteq U$ so $U = U_N$. Finally, for any $N' \geq N$, we have $U_N \subseteq U_{N'} \subseteq U = U_N$, i.e. $U_N = U_{N'}$. This means that the sequence is eventually constant. $\square$

Thus the Noetherian property is stronger than compactness. Compactness alone is not enough to guarantee that a game played with an infinite cover will end: one would like to take the fact that any open cover has a finite subcover and carry that finiteness into the gameplay, guaranteeing that the players form a cover after finitely many moves, but to invoke compactness one needs to have a cover already.

From a different point of view, we also wonder if and how we might actually embrace potentially infinite gameplay. The topological games in [6] and [7] are all built around infinite gameplay; a winner is not assessed by considering any one move but rather the infinite sequence of moves in its entirety. Could we rethink our game in that light, with a criterion for winning that is applied after a countable number of moves? The difficulty is that nim intuitively is a finite game: what does it mean to "take the last rock" if no player plays last?

3.2 Partitions of Covers

Thus far, we have considered arbitrary partitions. But we would like to know what ways of partitioning a cover already exist in the literature. What we have found so far entirely comes out of *Combinatorics of Open Covers* [6], and, though it is interesting, we are not yet sure how to relate it to our project. Some examples:

Definition 33. A space belongs to the class $\mathrm{Q}(\mathcal{A}, \mathcal{B})$ if for every partition of an open cover $\mathcal{U}$ from class $\mathcal{A}$ into countably many nonempty finite sets $\{\mathcal{F}_n\}$ there is a subcover $\mathcal{V}$ in $\mathcal{U}$ belonging to class $\mathcal{B}$ and which for each n has at most one element in common with $\mathcal{F}_n$.

Definition 34. For a space X, the class $\mathcal{A}$ of open covers of X is said to be *countably distinctly representable* by the class $\mathcal{B}$ of open covers, relative to the binary operation R, or $X \in \mathrm{CDR}_{\mathrm{R}}(\mathcal{A}, \mathcal{B})$, if for every sequence $(\mathcal{U}_n)$ of covers from $\mathcal{A}$ there is a sequence

(V_n) of covers from $\mathcal{B}$ such that

1. $\mathcal{V}_n \mathrm{R}\, \mathcal{U}_n$ for each n
2. $\mathcal{V}_n \cap \mathcal{V}_m = \emptyset$ when $n \neq m$

Observation 35. If $X \in \mathrm{CDR}_{\mathrm{Sub}}(\mathcal{A}, \mathcal{B})$, where B Sub A if B is a subcover of A, then every cover of X from $\mathcal{A}$ contains countably many disjoint covers from $\mathcal{B}$.

Definition 36. A space X is said to satisfy the partition relation $\mathcal{A} \rightarrow_{\Psi} (\mathcal{B})^2_2$ if for every cover $\mathcal{U}$ of X in class $\mathcal{A}$ and for every function $f : [\mathcal{U}]^2 \longrightarrow \{0, 1\}$ in class Ψ there is a subcover $\mathcal{V}$ of $\mathcal{U}$ in class $\mathcal{B}$ which is homogeneous for f, i.e. there is $i \in \{0, 1\}$ with $f([\mathcal{V}]^2) = i$.

3.3 Which nimber is a given cover game equivalent to?

We feel that a more direct engagement with combinatorial game theory could help illuminate strategies in our game. Our cover partition game is impartial, so it equivalent to a nimber according to the Sprague-Grundy Theorem. Are there tools in Sprague-Grundy theory that would provide a way to take a given cover and partition and find the associated nimber? Can we decompose our game into a sum of simpler games? Certainly there is more to explore here.

3.4 Topology

Though our project was motivated by topological games, our work so far has not specifically utilized much, if any, of the structure inherent in a topological space. We are not exactly sure how to go about doing this. The Noetherian idea presents one possibility, but there must be others.

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